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ON TRIANGLES WITH RATIONAL SIDES AND HAVING RATIONAL AREAS.

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In order to find an expression for the sides of a triangle with rational area we may use the formula:

$$\sqrt{\{s(s-a)(s-b)(s-c)\}} = T,$$

which shows that the condition for rationality is

$$s(s-a)(s-b)(s-c) = T^2,$$

a, b, c and T being rational numbers.

No three of the numbers s, $s-a$, $s-b$ and $s-c$ can have a common factor, if we restrict the sides of the triangle so that they shall have no factor in common. For any factor common to s, $s-a$, and $s-b$, say, would also be common to a, b and c. By this restriction we avoid to deal with triangles that are similar to any one fundamental triangle.

It has been proved, I believe, that $s = \frac{a+b+c}{2}$ is an integer for such triangles. However, it is easy to see that it must be an integer. For let $2s = p$, and by the formula

$$s(s-a)(s-b)(s-c) = T^2$$

we get

$$p(p-2a)(p-2b)(p-2c) = P^2, \text{ say,}$$

where p and P are integers. Hence

$$p^4 - p^3(2a+2b+2c) \equiv P^2 \pmod{4},$$

or

$$p^4 - p^3 \, . \, 2p \equiv -p^4 \equiv P^2 \pmod{4},$$

giving

$$p^4 + P^2 \equiv 0 \pmod{4}.$$

But this is an impossibility unless p and P are both even. That is, $p = 2s$, where s is an integer.

Any two of the numbers s, $s-a$, $s-b$ and $s-c$ may, however, have a common factor. Let us denote the H. C. F. to s and $s-a$ by α, to s and $s-b$ by β, to $s-c$ and $s-b$ by γ, to $s-c$ and $s-a$ by δ, to s and $s-c$ by ϵ, and to $s-a$ and $s-b$ by ζ. The factors of the product $s(s-a)(s-b)(s-c) = T^2$ thus assumed are:

$$\alpha^2\beta^2\gamma^2\delta^2\epsilon^2\zeta^2,$$

a square. The remaining factors must therefore also form a square; they are, moreover, prime to each other, and each must consequently be a square. Let us then denote the remaining factors of s, $s - a$, $s - b$ and $s - c$ by p^2, q^2, r^2, t^2 respectively.

We have now the equations:

$$\left.\begin{aligned} s &= \alpha\beta\varepsilon p^2 , \\ s - a &= \alpha\delta\zeta q^2 , \\ s - b &= \beta\gamma\zeta r^2 , \\ s - c &= \gamma\delta\varepsilon t^2 . \end{aligned}\right\} \qquad \text{I}$$

with the condition:

$$- \alpha\beta\varepsilon p^2 + \alpha\delta\zeta q^2 + \beta\gamma\zeta r^2 + \gamma\delta\varepsilon t^2 = 0 \qquad \text{II}$$

since:

$$- s + (s - a) + (s - b) + (s - c) = 0 .$$

It may here be noticed, that α is prime to r and t, β to q and t, γ to p and q, etc. For, if α were not prime to r, β would not be the H. C. F. to s and $s - b$; and so for the others.

The equations (I) and (II) will give expressions for the sides of the triangles having rational areas; the sides a, b, c being integers and freed from common factors.

Solving equation (II) for ζ we find:

$$\zeta = \varepsilon \cdot \frac{\alpha\beta p^2 - \gamma\delta t^2}{\alpha\delta q^2 + \beta\gamma r^2} .$$

Substituting this value for ζ in equations (1) and multiplying them by $\dfrac{\alpha\delta q^2 + \beta\gamma r^2}{\varepsilon}$ we obtain:

$$\left.\begin{aligned} s &= \alpha\beta p^2 (\alpha\delta q^2 + \beta\gamma r^2) \\ s - a &= \alpha\delta q^2 (\alpha\beta p^2 - \gamma\delta t^2) \\ s - b &= \beta\gamma r^2 (\alpha\beta p^2 - \gamma\delta t^2) \\ s - c &= \gamma\delta t^2 (\alpha\delta q^2 + \beta\gamma r^2) \end{aligned}\right\} \qquad \text{III}$$

Any three of these equations will give all triangles of the type sought by varying the arbitrary numbers α, β, γ, etc. But we must divide the right-hand members of the equations by the H. C. F. to:

$$\alpha\beta p^2 - \gamma\delta t^2 \text{ and } \alpha\delta q^2 + \beta\gamma r^2 .$$

Solving equations (III) for a, b and c we find:

$$\left.\begin{aligned} a &= \alpha\gamma\,(\beta^2p^2r^2 + \delta^2q^2t^2) \\ b &= \beta\delta\,(\alpha^2p^2q^2 + \gamma^2r^2t^2) \\ c &= (\alpha\delta q^2 + \beta\gamma r^2)\,(\alpha\beta p^2 - \gamma\delta t^2) \end{aligned}\right\}. \qquad \text{IV}$$

As the H. C. F. to $\alpha\delta q^2 + \beta\gamma r^2$ and $\alpha\beta p^2 - \gamma\delta t^2$ is common to all of the numbers s, $s - a$, $s - b$ and $s - c$ (see equations (III)), it is also common to a, b and c, and must be divided out from the right-hand members of equations (IV).

The sides a and b can have no common factor unless it be a factor of $\beta^2p^2r^2 + \delta^2q^2t^2$, since α, β, γ and δ are prime to each other, and $\alpha\gamma$ is prime to $\alpha^2p^2q^2 + \gamma^2r^2t^2$. For a similar reason a and c can have no factor in common unless it be a factor of $\beta^2p^2r^2 + \delta^2q^2t^2$. But this is the sum of two squares, prime to each other, and by a well-known theorem any factor of such a number is again the sum of two squares. We have thus the property of such triangles, that *if any two sides have a common factor, that factor must be the sum of two squares.*

The expressions in (IV) for the sides a, b, c may be put in a symmetrical form. Multiplying the right-hand members of the equations by $pqrt$, then putting A for αpq, B for βpr, C for γrt, and D for δqt, and we get the equations:

$$\left.\begin{aligned} a &= AC\,(B^2 + D^2) \\ b &= BD\,(A^2 + C^2) \\ c &= (BC + AD)\,(AB - CD) \end{aligned}\right\}. \qquad \text{V}$$

Multiply now by $p_1^2 + q_1^2$, and let

$$(B^2 + D^2)\,(p_1^2 + q_1^2) = (Bp_1 + Dq_1)^2 + (Dp_1 - Bq_1)^2 = r_1^2 + s_1^2, \text{ say};$$

$$(A^2 + C^2)\,(p_1^2 + q_1^2) = (Ap_1 - Cq_1)^2 + (Cp_1 + Aq_1)^2 = t_1^2 + w_1^2,$$

so that

$$Bp_1 + Dq_1 = r_1, \quad Dp_1 - Bq_1 = s_1, \quad Ap_1 - Cq_1 = t_1, \quad Cp_1 + Aq_1 = u_1.$$

If we now solve these equations for A, B, C and D, substitute in the equations given above (V) and multiply the right-hand members by $(p_1^2 + q_1^2)^3$, we obtain:

$$\left.\begin{aligned} a &= (p_1t_1 + q_1u_1)\,(p_1u_1 - q_1t_1)\,(r_1^2 + s_1^2), \\ b &= (p_1s_1 + q_1r_1)\,(p_1r_1 - q_1s_1)\,(t_1^2 + u_1^2), \\ c &= (r_1u_1 + s_1t_1)\,(r_1t_1 - s_1u_1)\,(p_1^2 + q_1^2). \end{aligned}\right\} \qquad \text{VI}$$

This result could also have been obtained by dividing out a supposed common factor to $B^2 + D^2$ and $A^2 + C^2$ from equations (V), which would also be a factor of $(BC + AD)\,(AB - CD)$. Let that factor be $p_1^2 + q_1^2$, and by a known proposition we see that we ought to have :

$$B = p_1 r_1 - q_1 s_1,\ D = p_1 s_1 + q_1 r_1,\ A = p_1 t_1 \pm q_1 u_1,\ C = p_1 u_1 \mp q_1 t_1,$$

assuming $B^2 + D^2 = (p_1^2 + q_1^2)(r_1^2 + s_1^2)$, and $A^2 + C^2 = (p_1^2 + q_1^2)(t_1^2 + u_1^2)$. By the substitution of these values for A, B, C and D, using the upper sign in A and C (in order that $p_1^2 + q_1^2$ may divide $(BC + AD)\,(AB - CD)$), we will get the equations (VI), the right-hand members multiplied by the assumed common factor $p_1^2 + q_1^2$.

By means of equations (V) we can get a very simple expression for the sides of the triangle.

If we divide the numbers expressing the sides by $ABCD$, we have :

$$a = \left[\frac{B}{D} + \frac{D}{B}\right],$$

$$b = \left[\frac{A}{C} + \frac{C}{A}\right],$$

$$c = \left[\frac{B}{D} + \frac{A}{C}\right]\left[1 - \frac{C}{A}\cdot\frac{D}{B}\right].$$

Let us now put $m = \dfrac{B}{D}$ and $n = \dfrac{A}{C}$, and the equations become :

$$\left.\begin{aligned} a &= m + \frac{1}{m}, \\ b &= n + \frac{1}{n}, \\ c &= (m + n)\left[1 - \frac{1}{m\,.\,n}\right] = m - \frac{1}{m} + n - \frac{1}{n}. \end{aligned}\right\} \quad \text{VII}$$

By giving to m and n all rational values—integral or fractional, positive or negative—we form all the triangles possible with rational sides and having rational areas.

If, by the preceding formulæ, one or more of the sides come out negative, we need merely to change their signs.